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# Stability of homomorphisms for a 3D Cauchy–Jensen type functional equation on $C^*$-ternary algebras

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**Abstract**

In this paper, we investigate homomorphisms between $C^*$-ternary algebras, and derivations on $C^*$-ternary algebras associated with the following Cauchy–Jensen type additive functional equation:

$$f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x+z}{2}+y\right)+f\left(\frac{y+z}{2}+x\right)=2\big(f(x)+f(y)+f(z)\big).$$



*Keywords:* Jensen functional equation; Homomorphism in quasi-Banach algebra; Hyers–Ulam–Rassias stability; $p$-Banach algebra

## 1. Introduction

The stability problem of functional equations originated from a question of Ulam [39] concerning the stability of group homomorphisms: Let $(G_1,*)$ be a group and let $(G_2,\diamond,d)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon>0$, does there exist $\delta(\epsilon)>0$ such that if a mapping $h:G_1\to G_2$ satisfies the inequality

$$d\big(h(x*y),h(x)\diamond h(y)\big)<\delta$$

for all $x,y\in G_1$, then there is a homomorphism $H:G_1\to G_2$ with

$$d\big(h(x),H(x)\big)<\epsilon$$

for all $x\in G_1$?

In other words, we are looking for situations when the homomorphisms are stable, i.e., if a mapping is almost a homomorphism, then there exists a true homomorphism near it. In 1941, Hyers [7] considered the case of approximately additive mappings in Banach spaces and satisfying the well-known weak Hyers inequality controlled by a positive constant.

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The famous Hyers stability result that appeared in [7] was generalized in the stability involving a sum of powers of norms by Aoki [2]. In 1978, Th.M. Rassias [32] provided a generalization of Hyers' Theorem which allows the *Cauchy difference* to be *unbounded*.

In 1982, J.M. Rassias [25] following the spirit of the innovative approach of Th.M. Rassias [32] for the unbounded Cauchy difference proved a similar stability theorem in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \cdot \|y\|^q$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$.

**Theorem 1.1** *(Th.M. Rassias). Let $f : E \to E'$ be a mapping from a normed vector space $E$ into a Banach space $E'$ subject to the inequality*

$$\big\| f(x+y) - f(x) - f(y) \big\| \leqslant \epsilon\big(\|x\|^p + \|y\|^p\big) \tag{1.1}$$

*for all $x, y \in E$, where $\epsilon$ and $p$ are constants with $\epsilon > 0$ and $p < 1$. Then the limit*

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

*exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies*

$$\big\| f(x) - L(x) \big\| \leqslant \frac{2\epsilon}{2 - 2^p} \|x\|^p \tag{1.2}$$

*for all $x \in E$. If $p < 0$ then inequality* (1.1) *holds for $x, y \neq 0$ and* (1.2) *for $x \neq 0$. Also, if the mapping $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then $L$ is $\mathbb{R}$-linear.*

**Theorem 1.2** *(J.M. Rassias). Let $X$ be a real normed linear space and $Y$ be a real complete normed linear space. Assume that $f : X \to Y$ is an approximately additive mapping for which there exist constants $\theta \geqslant 0$ and $p, q \in \mathbb{R}$ such that $r = p + q \neq 1$ and $f$ satisfies inequality*

$$\big\| f(x+y) - f(x) - f(y) \big\| \leqslant \theta \|x\|^p \|y\|^q$$

*for all $x, y \in X$. Then there exists a unique additive mapping $L : X \to Y$ satisfying*

$$\big\| f(x) - L(x) \big\| \leqslant \frac{\theta}{|2^r - 2|} \|x\|^r$$

*for all $x \in X$. If, in addition, $f : X \to Y$ is a mapping such that the transformation $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then $L$ is an $\mathbb{R}$-linear mapping.*

In 1990, Th.M. Rassias [33] during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geqslant 1$. In 1991, Z. Gajda [4] following the same approach as in Th.M. Rassias [32], gave an affirmative solution to this question for $p > 1$. It was shown by Z. Gajda [4], as well as by Th.M. Rassias and P. Šemrl [37] that one cannot prove a Th.M. Rassias' type theorem when $p = 1$. The counterexamples of Z. Gajda [4], as well as of Th.M. Rassias and P. Šemrl [37] have stimulated several mathematicians to invent new definitions of *approximately additive* or *approximately linear* mappings, cf. P. Găvruta [5], S. Jung [13], who among others studied the Hyers–Ulam–Rassias stability of functional equations. The inequality (1.1) that was introduced for the first time by Th.M. Rassias [32] provided a lot of influence in the development of a generalization of the Hyers–Ulam stability concept. This new concept is known as generalized *Hyers–Ulam stability* of functional equations (cf. the books of P. Czerwik [3], D.H. Hyers, G. Isac and Th.M. Rassias [8]).

In J.M. Rassias' Theorem, there was a singular case. Then for this singularity, a counterexample was given by Găvruta [6].

The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [9–11,14]). For further research developments in stability of functional equations, the readers are referred to the works of Park [15–24], J.M. Rassias [25–31], Th.M. Rassias [32–36], Skof [38] and the references cited therein.

A $C^*$-ternary algebra is a complex Banach space $A$, equipped with a ternary product $(x, y, z) \mapsto [x, y, z]$ of $A^3$ into $A$, which is $\mathbb{C}$-linear in the outer variables, conjugate $\mathbb{C}$-linear in the middle variable, and associative in the sense that $[x, y, [z, w, v]] = [x, [w, z, y], v] = [[x, y, z], w, v]$, and satisfies $\|[x, y, z]\| \leqslant \|x\| \cdot \|y\| \cdot \|z\|$

and $\|[x,x,x]\| = \|x\|^3$ (see [1,40]). Every left Hilbert $C^*$-module is a $C^*$-ternary algebra via the ternary product $[x,y,z] := \langle x,y\rangle z$.

If a $C^*$-ternary algebra $(A,[\cdot,\cdot,\cdot])$ has an identity, i.e., an element $e \in A$ such that $x = [x,e,e] = [e,e,x]$ for all $x \in A$, then it is a routine to verify that $A$, endowed with $x \circ y := [x,e,y]$ and $x^* := [e,x,e]$, is a unital $C^*$-algebra. Conversely, if $(A,\circ)$ is a unital $C^*$-algebra, then $[x,y,z] := x \circ y^* \circ z$ makes $A$ into a $C^*$-ternary algebra.

A $\mathbb{C}$-linear mapping $H : A \to B$ is called a *$C^*$-ternary algebra homomorphism* if

$$H\big([x,y,z]\big) = \big[H(x), H(y), H(z)\big]$$

for all $x,y,z \in A$. If, in addition, the mapping $H$ is bijective, then the mapping $H : A \to B$ is called a *$C^*$-ternary algebra isomorphism*. A $\mathbb{C}$-linear mapping $\delta : A \to A$ is called a *$C^*$-ternary derivation* if

$$\delta\big([x,y,z]\big) = \big[\delta(x), y, z\big] + \big[x, \delta(y), z\big] + \big[x, y, \delta(z)\big]$$

for all $x,y,z \in A$ (see [1]).

## 2. Stability of homomorphisms in $C^*$-ternary algebras

Throughout this section, assume that $A$ is a $C^*$-ternary algebra with norm $\|\cdot\|_A$ and that $B$ is a $C^*$-ternary algebra with norm $\|\cdot\|_B$.

We will use the following lemma in this paper.

**Lemma 2.1.** *Let $X$ and $Y$ be linear spaces and let $f : X \to Y$ be an additive mapping such that $f(\mu x) = \mu f(x)$ for all $x \in X$ and all $\mu \in \mathbb{T}^1$. Then the mapping $f$ is $\mathbb{C}$-linear.*

**Lemma 2.2.** *Let $X$ be a uniquely 2-divisible abelian group and $Y$ be a linear space. A mapping $f : X \to Y$ satisfies*

$$f\left(\frac{x+y}{2} + z\right) + f\left(\frac{x+z}{2} + y\right) + f\left(\frac{y+z}{2} + x\right) = 2\big[f(x) + f(y) + f(z)\big] \tag{2.1}$$

*for all $x,y,z \in X$ if and only if $f : X \to Y$ is additive.*

**Proof.** Suppose that $f$ satisfies (2.1). Letting $y = z = x$ in (2.1), we get $f(2x) = 2f(x)$ for all $x \in X$. So $f(0) = 0$ and $2f(x/2) = f(x)$ for all $x \in X$. Therefore by letting $y = -x$ and $z = 0$ in (2.1), we get $f(-x) = -f(x)$ for all $x \in X$. Letting $z = -y$ in (2.1), we get

$$f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) = f(x) \tag{2.2}$$

for all $x,y \in X$. Replacing $x$ and $y$ by $x+y$ and $x-y$ in (2.2), respectively, we infer that $f(x+y) = f(x) + f(y)$ for all $x,y \in X$. So the mapping $f : X \to Y$ is additive.

It is clear that each additive mapping satisfies (2.1). $\square$

For a given mapping $f : A \to B$, we define

$$\begin{aligned} Df(x,y,z) &:= f\left(\frac{x+y}{2} + z\right) + f\left(\frac{x+z}{2} + y\right) + f\left(\frac{y+z}{2} + x\right) - 2f(x) - 2f(y) - 2f(z), \\ D_\mu f(x,y,z) &:= f\left(\frac{\mu x+\mu y}{2} + \mu z\right) + f\left(\frac{\mu x+\mu z}{2} + \mu y\right) + f\left(\frac{\mu y+\mu z}{2} + \mu x\right) \\ &\quad - 2\mu f(x) - 2\mu f(y) - 2\mu f(z) \end{aligned}$$

for all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C}\colon |\lambda| = 1\}$ and all $x,y,z \in A$.

**Lemma 2.3.** *Let $X$ and $Y$ be linear spaces and let $f : X \to Y$ be a mapping such that*

$$D_\mu f(x,y,z) = 0 \tag{2.3}$$

*for all $\mu \in \mathbb{T}^1$ and all $x,y,z \in A$. Then the mapping $f : X \to Y$ is $\mathbb{C}$-linear.*

**Proof.** Letting $y = z = 0$ in (2.3) and using Lemma 2.2, we get $f(\mu x) = \mu f(x)$. Now by using Lemma 2.2 twice and Lemma 2.1, we infer that the mapping $f : X \to Y$ is $\mathbb{C}$-linear. $\square$

In the following we investigate the generalized Hyers–Ulam stability of (2.3).

**Theorem 2.4.** *Let $\varphi : A^3 \to [0,\infty)$ and $\psi : A^3 \to [0,\infty)$ be functions such that*

$$\tilde{\varphi}(x) := \sum_{n=0}^{\infty} \frac{1}{2^n}\varphi\bigl(2^n x, 2^n x, 2^n x\bigr) < \infty, \qquad \lim_{n\to\infty} \frac{1}{2^n}\varphi\bigl(2^n x, 2^n y, 2^n z\bigr) = 0, \tag{2.4}$$

$$\lim_{n\to\infty} \frac{1}{8^n}\psi\bigl(2^n x, 2^n y, 2^n z\bigr) = 0 \tag{2.5}$$

*for all $x, y, z \in A$. Suppose that $f : A \to B$ is a mapping satisfying*

$$\bigl\|D_\mu f(x, y, z)\bigr\|_B \leqslant \varphi(x, y, z), \tag{2.6}$$

$$\bigl\|f\bigl([x, y, z]\bigr) - \bigl[f(x), f(y), f(z)\bigr]\bigr\|_B \leqslant \psi(x, y, z) \tag{2.7}$$

*for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. Then there exists a unique $C^*$-ternary algebra homomorphism $H : A \to B$ such that*

$$\bigl\|f(x) - H(x)\bigr\|_B \leqslant \frac{1}{6}\tilde{\varphi}(x) \tag{2.8}$$

*for all $x \in A$.*

**Proof.** Letting $\mu = 1$ and $x = y = z$ in (2.6), we get

$$\bigl\|3f(2x) - 6f(x)\bigr\|_B \leqslant \varphi(x, x, x) \tag{2.9}$$

for all $x \in A$. If we replace $x$ by $2^n x$ in (2.9) and divide both sides of (2.9) by $3 \times 2^{n+1}$, we get

$$\left\|\frac{1}{2^{n+1}} f\bigl(2^{n+1}x\bigr) - \frac{1}{2^n} f\bigl(2^n x\bigr)\right\|_B \leqslant \frac{1}{3 \times 2^{n+1}}\varphi\bigl(2^n x, 2^n x, 2^n x\bigr)$$

for all $x \in A$ and all non-negative integers $n$. Hence

$$\begin{aligned}
\left\|\frac{1}{2^{n+1}} f\bigl(2^{n+1}x\bigr) - \frac{1}{2^m} f\bigl(2^m x\bigr)\right\|_B &= \left\|\sum_{k=m}^{n}\left[\frac{1}{2^{k+1}} f\bigl(2^{k+1}x\bigr) - \frac{1}{2^k} f\bigl(2^k x\bigr)\right]\right\|_B \\
&\leqslant \sum_{k=m}^{n}\left\|\frac{1}{2^{k+1}} f\bigl(2^{k+1}x\bigr) - \frac{1}{2^k} f\bigl(2^k x\bigr)\right\|_B \\
&\leqslant \frac{1}{6}\sum_{k=m}^{n}\frac{1}{2^k}\varphi\bigl(2^k x, 2^k x, 2^k x\bigr)
\end{aligned} \tag{2.10}$$

for all $x \in A$ and all non-negative integers $n \geqslant m \geqslant 0$. It follows from (2.4) and (2.10) that the sequence $\{\frac{1}{2^n} f(2^n x)\}$ is a Cauchy sequence in $B$ for all $x \in A$. Since $B$ is complete, the sequence $\{\frac{1}{2^n} f(2^n x)\}$ converges for all $x \in A$. Thus one can define the mapping $H : A \to B$ by

$$H(x) := \lim_{n\to\infty}\frac{1}{2^n} f\bigl(2^n x\bigr)$$

for all $x \in A$. Moreover, letting $m = 0$ and passing the limit $n \to \infty$ in (2.10) we get (2.8). It follows from (2.4) that

$$\begin{aligned}
\bigl\|D_\mu H(x, y, z)\bigr\|_B &= \lim_{n\to\infty}\frac{1}{2^n}\bigl\|D_\mu f\bigl(2^n x, 2^n y, 2^n z\bigr)\bigr\|_B \\
&\leqslant \lim_{n\to\infty}\frac{1}{2^n}\varphi\bigl(2^n x, 2^n y, 2^n z\bigr) = 0
\end{aligned}$$

for all $x, y, z \in A$. So $D_\mu H(x, y, z) = 0$ for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. By Lemma 2.3 the mapping $H : A \to B$ is $\mathbb{C}$-linear.

It follows from (2.5) and (2.7) that

$$\begin{aligned}\big\|H\big([x,y,z]\big) - \big[H(x), H(y), H(z)\big]\big\|_B &= \lim_{n\to\infty} \frac{1}{8^n} \big\|f\big([2^n x, 2^n y, 2^n z]\big) - \big[f\big(2^n x\big), f\big(2^n y\big), f\big(2^n z\big)\big]\big\|_B \\ &\leqslant \lim_{n\to\infty} \frac{1}{8^n} \psi\big(2^n x, 2^n y, 2^n z\big) = 0\end{aligned}$$

for all $x, y, z \in A$. Therefore

$$H\big([x,y,z]\big) = \big[H(x), H(y), H(z)\big]$$

for all $x, y, z \in A$. Therefore the mapping $H : A \to B$ is a $C^*$-ternary algebra homomorphism.

Now, let $I : A \to B$ be another $C^*$-ternary algebra homomorphism satisfying (2.8). Then we have from (2.4) that

$$\begin{aligned}\big\|H(x) - I(x)\big\|_B &= \lim_{n\to\infty} \frac{1}{2^n} \big\|f\big(2^n x\big) - I\big(2^n x\big)\big\|_B \\ &\leqslant \frac{1}{6} \lim_{n\to\infty} \frac{1}{2^n} \tilde{\varphi}\big(2^n x\big) \\ &= \frac{1}{6} \lim_{n\to\infty} \sum_{k=n}^{\infty} \frac{1}{2^k} \varphi\big(2^k x, 2^k x, 2^k x\big) = 0\end{aligned}$$

for all $x \in A$. So $H(x) = I(x)$ for all $x \in A$. This proves the uniqueness of $H$. Thus the mapping $H : A \to B$ is a unique $C^*$-ternary algebra homomorphism satisfying (2.8). $\square$

**Corollary 2.5.** *Let $\epsilon, \theta, p_1, p_2, p_3, q_1, q_2, q_3$ be positive real numbers such that $p_1, p_2, p_3 < 1$ and $q_1, q_2, q_3 < 3$. Suppose that $f : A \to B$ is a mapping satisfying*

$$\big\|D_\mu f(x,y,z)\big\|_B \leqslant \theta\big(\|x\|_A^{p_1} + \|y\|_A^{p_2} + \|z\|_A^{p_3}\big), \tag{2.11}$$

$$\big\|f\big([x,y,z]\big) - \big[f(x), f(y), f(z)\big]\big\|_B \leqslant \epsilon\big(\|x\|_A^{q_1} + \|y\|_A^{q_2} + \|z\|_A^{q_3}\big) \tag{2.12}$$

*for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. Then there exists a unique $C^*$-ternary algebra homomorphism $H : A \to B$ such that*

$$\big\|f(x) - H(x)\big\|_B \leqslant \frac{\theta}{3}\left\{\frac{1}{2 - 2^{p_1}}\|x\|_A^{p_1} + \frac{1}{2 - 2^{p_2}}\|x\|_A^{p_2} + \frac{1}{2 - 2^{p_3}}\|x\|_A^{p_3}\right\} \tag{2.13}$$

*for all $x \in A$.*

**Remark 2.6.** Replacing (2.11) by $\|Df(x,y,z)\|_B \leqslant \theta(\|x\|_A^{p_1} + \|y\|_A^{p_2} + \|z\|_A^{p_3})$, in Corollary 2.5, we get that the mapping $H : A \to B$ is additive and satisfies (2.13). By using the results of [12,37], we prove in the following example that the mapping constructed by Rassias and Šemrl serves as a counterexample for the case $p_1 = p_2 = p_3 = 1$.

**Example 2.7.** We prove that the continuous real-valued mapping defined by

$$f(x) = \begin{cases} x \log_2(x+1), & x \geqslant 0, \\ x \log_2|x-1|, & x < 0, \end{cases}$$

satisfies the inequality

$$\big|Df(x,y,z)\big| \leqslant 4\big(|x| + |y| + |z|\big)$$

for all $x, y, z \in \mathbb{R}$, and the range of $|f(x) - H(x)|/|x|$ for $x \neq 0$ is unbounded for each additive mapping $H : \mathbb{R} \to \mathbb{R}$.

It follows from [12,37] that the mapping $f$ satisfies the following inequalities:

$$\begin{aligned}&\big|f(x+y) - f(x) - f(y)\big| \leqslant |x| + |y|, \\ &\left|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right| \leqslant 2\big(|x| + |y|\big)\end{aligned}$$

for all $x, y \in \mathbb{R}$. Therefore we have

$$\begin{aligned}
\left|Df(x,y,z)\right| &\leqslant \left|f\left(\frac{x+y}{2}+z\right)-f\left(\frac{x+y}{2}\right)-f(z)\right|+\frac{1}{2}\left|2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right| \\
&\quad+\left|f\left(\frac{x+z}{2}+y\right)-f\left(\frac{x+z}{2}\right)-f(y)\right|+\frac{1}{2}\left|2f\left(\frac{x+z}{2}\right)-f(x)-f(z)\right| \\
&\quad+\left|f\left(\frac{y+z}{2}+x\right)-f\left(\frac{y+z}{2}\right)-f(x)\right|+\frac{1}{2}\left|2f\left(\frac{y+z}{2}\right)-f(y)-f(z)\right| \\
&\leqslant 4\big(|x|+|y|+|z|\big)
\end{aligned}$$

for all $x, y, z \in \mathbb{R}$. Since $\lim_{x\to\infty}\frac{f(x)}{x}=+\infty$, then the range of $|f(x)-H(x)|/|x|$ for $x \neq 0$ is unbounded for each additive mapping $H:\mathbb{R}\to\mathbb{R}$.

**Theorem 2.8.** *Let $\Phi : A^3 \to [0,\infty)$ and $\Psi : A^3 \to [0,\infty)$ be functions such that*

$$\widetilde{\Phi}(x) := \sum_{n=1}^{\infty} 2^n \Phi\left(\frac{x}{2^n},\frac{x}{2^n},\frac{x}{2^n}\right) < \infty, \qquad \lim_{n\to\infty} 2^n \Phi\left(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\right) = 0, \tag{2.14}$$

$$\lim_{n\to\infty} 8^n \Psi\left(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\right) = 0 \tag{2.15}$$

*for all $x, y, z \in A$. Suppose that $f : A \to B$ is a mapping satisfying*

$$\left\|D_\mu f(x,y,z)\right\|_B \leqslant \Phi(x,y,z), \tag{2.16}$$

$$\left\|f\big([x,y,z]\big)-\big[f(x),f(y),f(z)\big]\right\|_B \leqslant \Psi(x,y,z) \tag{2.17}$$

*for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. Then there exists a unique $C^*$-ternary algebra homomorphism $H : A \to B$ such that*

$$\left\|f(x)-H(x)\right\|_B \leqslant \frac{1}{6}\widetilde{\Phi}(x) \tag{2.18}$$

*for all $x \in A$.*

**Proof.** Letting $\mu = 1$ and $x = y = z$ in (2.16), we get

$$\left\|f(2x)-2f(x)\right\|_B \leqslant \frac{1}{3}\Phi(x,x,x) \tag{2.19}$$

for all $x \in A$. If we replace $x$ by $\frac{x}{2^{n+1}}$ in (2.19) and multiply both sides of (2.19) to $2^n$, we get

$$\left\|2^{n+1}f\left(\frac{x}{2^{n+1}}\right)-2^n f\left(\frac{x}{2^n}\right)\right\|_B \leqslant \frac{2^n}{3}\Phi\left(\frac{x}{2^{n+1}},\frac{x}{2^{n+1}},\frac{x}{2^{n+1}}\right)$$

for all $x \in A$ and all non-negative integers $n$. Hence

$$\begin{aligned}
\left\|2^{n+1}f\left(\frac{x}{2^{n+1}}\right)-2^m f\left(\frac{x}{2^m}\right)\right\|_B &= \left\|\sum_{k=m}^{n}\left[2^{k+1}f\left(\frac{x}{2^{k+1}}\right)-2^k f\left(\frac{x}{2^k}\right)\right]\right\|_B \\
&\leqslant \sum_{k=m}^{n}\left\|2^{k+1}f\left(\frac{x}{2^{k+1}}\right)-2^k f\left(\frac{x}{2^k}\right)\right\|_B \\
&\leqslant \frac{1}{6}\sum_{k=m}^{n} 2^{k+1}\Phi\left(\frac{x}{2^{k+1}},\frac{x}{2^{k+1}},\frac{x}{2^{k+1}}\right)
\end{aligned} \tag{2.20}$$

for all $x \in A$ and all non-negative integers $n \geqslant m \geqslant 0$. It follows from (2.14) and (2.20) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence in $B$ for all $x \in A$. Since $B$ is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges for all $x \in A$. Thus one can define the mapping $H : A \to B$ by

$$H(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in A$. Moreover, letting $m = 0$ and passing the limit $n \to \infty$ in (2.20) we get (2.18). The rest of the proof is similar to the proof of Theorem 2.4. □

**Corollary 2.9.** *Let $\epsilon, \theta, p_1, p_2, p_3, q_1, q_2$ and $q_3$ be non-negative real numbers such that $p_1, p_2, p_3 > 1$ and $q_1, q_2, q_3 > 3$. Suppose that $f : A \to B$ is a mapping satisfying* (2.11) *and* (2.12). *Then there exists a unique $C^*$-ternary algebra homomorphism $H : A \to B$ such that*

$$\left\| f(x) - H(x)\right\|_B \leqslant \frac{\theta}{3}\left\{\frac{1}{2^{p_1}-2}\|x\|_A^{p_1} + \frac{1}{2^{p_2}-2}\|x\|_A^{p_2} + \frac{1}{2^{p_3}-2}\|x\|_A^{p_3}\right\}$$

*for all $x \in A$.*

## 3. Homomorphisms between $C^*$-ternary algebras

In the following we investigate the generalized Hyers–Ulam stability of (2.3).

**Lemma 3.1.** *Let $X$ and $Y$ be linear spaces. A mapping $f : X \to Y$ satisfies* (2.1) *for all $x, y, z \in X \setminus \{0\}$ if and only if $f : X \to Y$ is additive.*

**Proof.** Suppose that $f$ satisfies (2.1). Letting $y = z = x$ in (2.1), we get

$$f(2x) = 2f(x) \tag{3.1}$$

for all $x \in X \setminus \{0\}$. Letting $y = z = -x$ in (2.1), we get

$$2f(-x) + 2f(x) = f(0) \tag{3.2}$$

for all $x \in X \setminus \{0\}$. Letting $y = 3x, z = -x$ in (2.1) and using (3.1), we get

$$f(3x) = f(x) - 2f(-x) \tag{3.3}$$

for all $x \in X \setminus \{0\}$. It follows from (3.1) that $2f(x/2) = f(x)$ for all $x \in X \setminus \{0\}$. So by letting $y = x$ and $z = 2x$ in (2.1) and using (3.1), we get

$$f(5x) + f(3x) = 8f(x) \tag{3.4}$$

for all $x \in X \setminus \{0\}$. Putting $y = 5x$ and $z = -x$ in (2.1) and using (3.2), we get

$$f(5x) - f(3x) = 2f(x) - f(0) \tag{3.5}$$

for all $x \in X \setminus \{0\}$. It follows from (3.4) and (3.5) that

$$2f(3x) = 6f(x) + f(0) \tag{3.6}$$

for all $x \in X \setminus \{0\}$. It follows from (3.3) and (3.6) that

$$4\left[f(x) + f(-x)\right] + f(0) = 0 \tag{3.7}$$

for all $x \in X \setminus \{0\}$. It follows from (3.2) and (3.7) that $f(0) = 0$. Hence it follows from (3.2) that $f$ is odd. Therefore by letting $z = -x$ in (2.1), we get

$$f\left(\frac{x+y}{2}\right) + f\left(\frac{y-x}{2}\right) = f(y) \tag{3.8}$$

for all $x, y \in X \setminus \{0\}$. Since $f$ is odd, then (3.8) holds for all $x, y \in X$. Replacing $x$ and $y$ by $x-y$ and $x+y$ in (3.8), respectively, we get $f(x+y) = f(x) + f(y)$ for all $x, y \in X$. So the mapping $f : X \to Y$ is additive.

It is clear that each additive mapping satisfies (2.1). □

**Notation.** Let $X$ be a linear space. $x \in X^\star$ means $x \in X$ or $x \in X \setminus \{0\}$.

**Theorem 3.2.** *Let $\epsilon, \theta$ be non-negative real numbers and let $p_1, p_2, p_3, q_1, q_2, q_3$ be real numbers such that $p_i < 0$ for all $1 \leqslant i \leqslant 3$ and $q_j \neq 1$ for some $1 \leqslant j \leqslant 3$. Suppose that $f : A \to B$ is a mapping satisfying*

$$\left\| D_\mu f(x,y,z) \right\|_B \leqslant \theta \|x\|_A^{p_1} \|y\|_A^{p_2} \|z\|_A^{p_3}, \tag{3.9}$$

$$\left\| f\big([x,y,z]\big) - \big[f(x), f(y), f(z)\big] \right\|_B \leqslant \epsilon \|x\|_A^{q_1} \|y\|_A^{q_2} \|z\|_A^{q_3} \tag{3.10}$$

*for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A^\star$. Then there exists a unique $C^*$-ternary algebra homomorphism $H : A \to B$ such that*

$$\left\| f(x) - H(x) \right\|_B \leqslant \frac{1}{3(2-2^\lambda)} \|x\|_A^\lambda \tag{3.11}$$

*for all $x \in A \setminus \{0\}$, where $\lambda = p_1 + p_2 + p_3$.*

**Proof.** Letting $\mu = 1$ and $x = y = z$ in (3.9), we get

$$\left\| 3f(2x) - 6f(x) \right\|_B \leqslant \|x\|_A^\lambda \tag{3.12}$$

for all $x \in A \setminus \{0\}$. If we replace $x$ by $2^n x$ in (3.12) and divide both sides of (3.12) by $6 \times 2^n$, we get

$$\left\| \frac{1}{2^{n+1}} f\big(2^{n+1}x\big) - \frac{1}{2^n} f\big(2^n x\big) \right\|_B \leqslant \frac{1}{6} \left( \frac{2^\lambda}{2} \right)^n \|x\|_A^\lambda$$

for all $x \in A \setminus \{0\}$ and all non-negative integers $n$. Hence

$$\begin{aligned}
\left\| \frac{1}{2^{n+1}} f\big(2^{n+1}x\big) - \frac{1}{2^m} f\big(2^m x\big) \right\|_B &= \left\| \sum_{k=m}^{n} \left[ \frac{1}{2^{k+1}} f\big(2^{k+1}x\big) - \frac{1}{2^k} f\big(2^k x\big) \right] \right\|_B \\
&\leqslant \sum_{k=m}^{n} \left\| \frac{1}{2^{k+1}} f\big(2^{k+1}x\big) - \frac{1}{2^k} f\big(2^k x\big) \right\|_B \\
&\leqslant \frac{1}{6} \sum_{k=m}^{n} \left( \frac{2^\lambda}{2} \right)^k \|x\|_A^\lambda
\end{aligned} \tag{3.13}$$

for all $x \in A \setminus \{0\}$ and all non-negative integers $n \geqslant m \geqslant 0$. Since $\lambda < 0$, it follows from (3.13) that the sequence $\{\frac{1}{2^n} f(2^n x)\}$ is a Cauchy sequence in $B$ for all $x \in A$. Since $B$ is complete, the sequence $\{\frac{1}{2^n} f(2^n x)\}$ converges for all $x \in A$. Thus one can define the mapping $H : A \to B$ by

$$H(x) := \lim_{n \to \infty} \frac{1}{2^n} f\big(2^n x\big)$$

for all $x \in A$. Moreover, letting $m = 0$ and passing the limit $n \to \infty$ in (3.13) we get (3.11). It follows from (2.4) that

$$\begin{aligned}
\left\| D_\mu H(x,y,z) \right\|_B &= \lim_{n \to \infty} \frac{1}{2^n} \left\| D_\mu f\big(2^n x, 2^n y, 2^n z\big) \right\|_B \\
&\leqslant \lim_{n \to \infty} \left( \frac{2^\lambda}{2} \right)^n \|x\|_A^{p_1} \|y\|_A^{p_2} \|z\|_A^{p_3} = 0
\end{aligned}$$

for all $x, y, z \in A \setminus \{0\}$. So $D_\mu H(x,y,z) = 0$ for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in \setminus \{0\}$. By Lemmas 3.1 and 2.3 the mapping $H : A \to B$ is $\mathbb{C}$-linear.

Without any loss of generality, we may suppose that $q_1 \neq 1$. Let $q_1 > 1$. It follows from (3.10) that

$$\begin{aligned}\big\|H\big([x,y,z]\big)-\big[H(x),H(y),H(z)\big]\big\|_B&=\lim_{n\to\infty}2^n\left\|f\left(\left[\frac{x}{2^n},y,z\right]\right)-\left[f\left(\frac{x}{2^n}\right),f(y),f(z)\right]\right\|_B\\&\leqslant\epsilon\lim_{n\to\infty}\frac{2^n}{2^{nq_1}}\|x\|_A^{q_1}\|y\|_A^{q_2}\|z\|_A^{q_3}=0\end{aligned}$$

for all $x,y,z\in A^\star$. Therefore

$$H\big([x,y,z]\big)=\big[H(x),H(y),H(z)\big] \tag{3.14}$$

for all $x,y,z\in A^\star$. Since $H(0)=0$, then (3.14) holds for all $x,y,z\in A$. Similarly, for $q_1<1$, we get (3.14). So the mapping $H:A\to B$ is a $C^*$-ternary algebra homomorphism.

Now, let $T:A\to B$ be another $C^*$-ternary algebra homomorphism satisfying (3.11). Then we have from (2.4) that

$$\begin{aligned}\big\|H(x)-T(x)\big\|_B&=\lim_{n\to\infty}\frac{1}{2^n}\big\|f\big(2^nx\big)-T\big(2^nx\big)\big\|_B\\&\leqslant\frac{1}{3(2-2^\lambda)}\lim_{n\to\infty}\left(\frac{2^\lambda}{2}\right)^n\|x\|_A^\lambda=0\end{aligned}$$

for all $x\in A\setminus\{0\}$. Since $H(0)=T(0)=0$, so $H(x)=T(x)$ for all $x\in A$. This proves the uniqueness of $H$. Thus the mapping $H:A\to B$ is a unique $C^*$-ternary algebra homomorphism satisfying (3.11). □

**Remark 3.3.** Theorem 3.2 will be valid if we replace the condition $q_j\neq 1$ for some $1\leqslant j\leqslant 3$ by one of the conditions $q_1+q_2+q_3\neq 3$ or $q_i+q_j\neq 2$ for some $1\leqslant i<j\leqslant 3$.

**Theorem 3.4.** *Let $q_1,q_2,q_3$ be real numbers and $\epsilon,\theta,p_1,p_2,p_3$ be non-negative real numbers such that $p_i>0$ and $q_j\neq 1$ for some $1\leqslant i,j\leqslant 3$. Suppose that $f:A\to B$ is a mapping satisfying* (3.9) *and* (3.10) *for all $\mu\in\mathbb{T}^1$ and all $x,y,z\in A$ ($x,y,z\in A\setminus\{0\}$ when $q_i<0$ for some $1\leqslant i\leqslant 3$). Then the mapping $f:A\to B$ is a $C^*$-ternary algebra homomorphism.*

**Proof.** Without any loss of generality, we suppose $p_1>0$. By letting $x=y=z=0$ in (3.9), we get $f(0)=0$. Letting $x=y=0$ and replacing $z$ by $2z$ in (3.9), we get

$$f(2\mu z)+2f(\mu z)=2\mu f(2z) \tag{3.15}$$

for all $\mu\in\mathbb{T}^1$ and all $z\in A$. Letting $\mu=1$ in (3.15), we get

$$f(2z)=2f(z) \tag{3.16}$$

for all $z\in A$. We get from (3.15) and (3.16) that $f(\mu z)=\mu f(z)$ for all $\mu\in\mathbb{T}^1$ and all $z\in A$. Therefore $f$ is an odd function.

Letting $x=0$ and replacing $y$ and $z$ by $2y$ and $2z$ in (3.9), respectively, we get

$$f(y+2z)+f(z+2y)+f(y+z)=4f(y)+4f(z) \tag{3.17}$$

for all $y,z\in A$. Replacing $y$ by $y+z$ and $z$ by $-z$ in (3.17) and using the oddness of $f$, we get

$$f(y-z)+f(2y+z)+f(y)=4f(y+z)-4f(z) \tag{3.18}$$

for all $y,z\in A$. Replacing $y$ by $z$ and $z$ by $y$ in (3.18) and using the oddness of $f$, we get

$$-f(y-z)+f(2z+y)+f(z)=4f(y+z)-4f(y) \tag{3.19}$$

for all $y,z\in A$. Adding (3.18) to (3.19) we have

$$f(y+2z)+f(z+2y)=8f(y+z)-5f(y)-5f(z) \tag{3.20}$$

for all $y,z\in A$. Now, by (3.17) and (3.20), we have $f(y+z)=f(y)+f(z)$ for all $y,z\in A$. Hence by Lemma 2.1 the mapping $f:A\to B$ is $\mathbb{C}$-linear.

Without any loss of generality, we may suppose that $q_1 \neq 1$. Let $q_1 > 1$. It follows from (3.10) that

$$\begin{aligned} \big\| f\big([x,y,z]\big) - \big[f(x), f(y), f(z)\big]\big\|_B &= \lim_{n\to\infty} 2^n \left\| f\left(\left[\frac{x}{2^n}, y, z\right]\right) - \left[f\left(\frac{x}{2^n}\right), f(y), f(z)\right]\right\|_B \\ &\leqslant \epsilon \lim_{n\to\infty} \frac{2^n}{2^{nq_1}} \|x\|_A^{q_1} \|y\|_A^{q_2} \|z\|_A^{q_3} = 0 \end{aligned}$$

for all $x, y, z \in A$. Therefore

$$f\big([x,y,z]\big) = \big[f(x), f(y), f(z)\big] \tag{3.21}$$

for all $x, y, z \in A$ ($x, y, z \in A \setminus \{0\}$ when $q_i < 0$ for some $2 \leqslant i \leqslant 3$). Since $f(0) = 0$, then (3.21) holds for all $x, y, z \in A$ when $q_i < 0$ for some $2 \leqslant i \leqslant 3$. Similarly, for $q_1 < 1$, we get (3.21). So the mapping $f : A \to B$ is a $C^*$-ternary algebra homomorphism. □

We will use the following lemma in the proof of the next theorem.

**Lemma 3.5.** *Let $X$ and $Y$ be linear spaces. An odd mapping $f : X \to Y$ satisfies*

$$f\left(\frac{x+y}{2}\right) + f\left(\frac{x}{2} + y\right) + f\left(\frac{y}{2} + x\right) = 2\big[f(x) + f(y)\big] \tag{3.22}$$

*for all $x, y \in X \setminus \{0\}$ if and only if $f : X \to Y$ is additive.*

**Proof.** Suppose that $f$ satisfies (3.22). Since $f$ is odd, then $f(0) = 0$. Letting $y = x$ in (3.22), we get

$$f\left(\frac{3x}{2}\right) = \frac{3}{2} f(x) \tag{3.23}$$

for all $x \in X \setminus \{0\}$. Letting $y = 2x$ in (3.22) and using (3.23), we get

$$f\left(\frac{5x}{2}\right) = f(2x) + \frac{1}{2} f(x) \tag{3.24}$$

for all $x \in X \setminus \{0\}$. Letting $y = -2x$ in (3.22) and using the oddness of $f$, we get

$$f\left(\frac{3x}{2}\right) + f\left(\frac{x}{2}\right) = 2f(2x) - 2f(x) \tag{3.25}$$

for all $x \in X \setminus \{0\}$. It follows from (3.25) that

$$f(3x) + f(x) = 2f(4x) - 2f(2x) \tag{3.26}$$

for all $x \in X \setminus \{0\}$. Letting $y = 4x$ in (3.22) and using (3.23) and (3.24), we get

$$5f(3x) = 4f(4x) - 2f(2x) + 3f(x) \tag{3.27}$$

for all $x \in X \setminus \{0\}$. It follows from (3.26) and (3.27) that

$$3f(4x) = 4f(2x) + 4f(x) \tag{3.28}$$

for all $x \in X \setminus \{0\}$. It follows from (3.23) and (3.25) that

$$7f(x) + 2f\left(\frac{x}{2}\right) = 4f(2x)$$

for all $x \in X \setminus \{0\}$. Replacing $x$ by $2x$ in the last equation, we get

$$4f(4x) = 7f(2x) + 2f(x) \tag{3.29}$$

for all $x \in X \setminus \{0\}$. It follows from (3.28) and (3.29) that $f(2x) = 2f(x)$ for all $x \in X \setminus \{0\}$. Since $f(0) = 0$, then $f(2x) = 2f(x)$ for all $x \in X$. Therefore (3.22) holds for all $x, y \in X$. Hence the mapping $f$ satisfies (3.17) for all $y, z \in X$. Using the proof of Theorem 3.4, we get that the mapping $f : X \to Y$ is additive.

It is clear that each additive mapping satisfies (3.22). □

**Theorem 3.6.** *Let $\epsilon, \theta$ be non-negative real numbers and let $p_1, p_2, p_3, q_1, q_2, q_3$ be real numbers such that $p_i p_j < 0$ for some $1 \leqslant i < j \leqslant 3$ and $q_j \neq 1$ for some $1 \leqslant j \leqslant 3$. Suppose that $f : A \to B$ is a mapping satisfying* (3.9) *and* (3.10) *for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A^\star$. Then the mapping $f : A \to B$ is a $C^*$-ternary algebra homomorphism.*

**Proof.** Without any loss of generality, we may assume that $p_3 > 0$. Let $\mu = 1$. Letting $z = 0$ in (3.9), we get

$$f\left(\frac{x+y}{2}\right) + f\left(\frac{x}{2} + y\right) + f\left(\frac{y}{2} + x\right) = 2\big[f(x) + f(y) + f(0)\big] \tag{3.30}$$

for all $x, y \in A \setminus \{0\}$. We show that $f$ is additive.

Letting $y = -x$ in (3.30), we get

$$f\left(\frac{x}{2}\right) + f\left(\frac{-x}{2}\right) = 2\big[f(x) + f(-x)\big] + f(0) \tag{3.31}$$

for all $x \in A \setminus \{0\}$. It follows from (3.31) that

$$f(x) + f(-x) = 2\big[f(2x) + f(-2x)\big] + f(0), \tag{3.32}$$

$$f\left(\frac{3x}{2}\right) + f\left(\frac{-3x}{2}\right) = 2\big[f(3x) + f(-3x)\big] + f(0) \tag{3.33}$$

for all $x \in A \setminus \{0\}$. Letting $y = x$ in (3.30), we get

$$2f\left(\frac{3x}{2}\right) = 3f(x) + 2f(0) \tag{3.34}$$

for all $x \in A \setminus \{0\}$. It follows from (3.34) that

$$2\left[f\left(\frac{3x}{2}\right) + f\left(\frac{-3x}{2}\right)\right] = 3\big[f(x) + f(-x)\big] + 4f(0), \tag{3.35}$$

$$2\big[f(3x) + f(-3x)\big] = 3\big[f(2x) + f(-2x)\big] + 4f(0) \tag{3.36}$$

for all $x \in A \setminus \{0\}$. It follows from (3.33) and (3.35) that

$$3\big[f(x) + f(-x)\big] + 2f(0) = 4\big[f(3x) + f(-3x)\big] \tag{3.37}$$

for all $x \in A \setminus \{0\}$. It follows from (3.36) and (3.37) that

$$f(x) + f(-x) = 2\big[f(2x) + f(-2x) + f(0)\big] \tag{3.38}$$

for all $x \in A \setminus \{0\}$. Now, we get from (3.32) and (3.38) that $f(0) = 0$. Hence (3.38) implies that

$$f(x) + f(-x) = 2\big[f(2x) + f(-2x)\big] \tag{3.39}$$

for all $x \in A \setminus \{0\}$. Letting $y = -2x$ in (3.30) and using (3.34) (with $f(0) = 0$), we get

$$f\left(\frac{-x}{2}\right) + \frac{3}{2}f(-x) = 2\big[f(x) + f(-2x)\big]$$

for all $x \in A \setminus \{0\}$. It follows from the last equation that

$$\left[f\left(\frac{x}{2}\right) + f\left(\frac{-x}{2}\right)\right] + \frac{3}{2}\big[f(x) + f(-x)\big] = 2\big[f(x) + f(-x)\big] + 2\big[f(2x) + f(-2x)\big] \tag{3.40}$$

for all $x \in A \setminus \{0\}$. Since $f(0) = 0$, then it follows from (3.31), (3.39) and (3.40) that $f(-x) = -f(x)$ for all $x \in A \setminus \{0\}$. Since $f(0) = 0$, then $f$ is odd. Therefore the odd mapping $f : A \to B$ satisfies (3.22) for all $x, y \in A \setminus \{0\}$. So by Lemma 3.5, the mapping $f$ is additive. Therefore by letting $z = 0$ and $y = x$ in (3.9), we get $f(\mu x) = \mu f(x)$ for all $x \in A \setminus \{0\}$. Since $f(0) = 0$, then $f(\mu x) = \mu f(x)$ for all $x \in A$. So by Lemma 2.1, the mapping $f$ is $\mathbb{C}$-linear.

The rest of the proof is similar to the proof of Theorem 3.4. □

**Theorem 3.7.** *Let $q_1, q_2, q_3$ be real numbers and let $\epsilon, \theta, p_1, p_2, p_3$ be non-negative real numbers such that $q_1 + q_2 + q_3 \neq 3$ and $p_i > 0$ for some $1 \leqslant i \leqslant 3$. Suppose that $f : A \to B$ is a mapping satisfying* (3.9) *and* (3.10) *for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$ ($x, y, z \in A \setminus \{0\}$ when $q_i < 0$ for some $1 \leqslant i \leqslant 3$). Then the mapping $f : A \to B$ is a $C^*$-ternary algebra homomorphism.*

**Proof.** Similarly to the proof of Theorem 3.4, the mapping $f : A \to B$ is $\mathbb{C}$-linear. Let $q_1 + q_2 + q_3 > 3$. It follows from (3.10) that

$$\begin{aligned}\big\|f\big([x,y,z]\big) - \big[f(x), f(y), f(z)\big]\big\|_B &= \lim_{n\to\infty} 8^n \left\| f\left(\left[\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right]\right) - \left[f\left(\frac{x}{2^n}\right), f\left(\frac{y}{2^n}\right), f\left(\frac{z}{2^n}\right)\right]\right\|_B \\ &\leqslant \epsilon \lim_{n\to\infty} \frac{8^n}{2^{n(q_1+q_2+q_3)}} \|x\|_A^{q_1} \|y\|_A^{q_2} \|z\|_A^{q_3} = 0\end{aligned}$$

for all $x, y, z \in A$. Therefore we get (3.21) for all $x, y, z \in A$ ($x, y, z \in A \setminus \{0\}$ when $q_i < 0$ for some $1 \leqslant i \leqslant 3$). Since $f(0) = 0$, then (3.21) holds for all $x, y, z \in A$ when $q_i < 0$ for some $1 \leqslant i \leqslant 3$. Similarly, for $q_1 + q_2 + q_3 < 3$, we get (3.21). So the mapping $f : A \to B$ is a $C^*$-ternary algebra homomorphism. □

**Remark 3.8.** If we replace the condition $q_1 + q_2 + q_3 \neq 3$ in Theorem 3.7 by $q_i + q_j \neq 2$ for some $1 \leqslant i < j \leqslant 3$, then by using the similar proof of Theorem 3.7, we get that the mapping $f : A \to B$ is a $C^*$-ternary algebra homomorphism.

**Remark 3.9.** It is an open problem: can we prove Theorems 3.2, 3.4, 3.6 and 3.7 when $q_1 = q_2 = q_3 = 1$?

## 4. Homomorphisms between unital $C^*$-ternary algebras

Throughout this section, assume that $A$ is a unital $C^*$-ternary algebra with norm $\|\cdot\|_A$, unit $e$ and that $B$ is a $C^*$-ternary algebra with norm $\|\cdot\|_B$ and unit $e'$.

We investigate homomorphisms between unital $C^*$-ternary algebras, associated to the functional equation $D_\mu f(x, y, z) = 0$.

**Theorem 4.1.** *Let $\epsilon, \theta, p_1, p_2, p_3, q_1, q_2, q_3$ be positive real numbers such that $p_1, p_2, p_3 < 1$, $q_1, q_2 < 2$ and $q_3 < 3$. Suppose that $f : A \to B$ is a mapping satisfying* (2.11) *and* (2.12). *If there exists a real number $\lambda > 1$ ($0 < \lambda < 1$) and an element $x_0 \in A$ such that $\lim_{n\to\infty} \frac{1}{\lambda^n} f(\lambda^n x_0) = e'$ ($\lim_{n\to\infty} \lambda^n f(\frac{x_0}{\lambda^n}) = e'$), then the mapping $f : A \to B$ is a $C^*$-ternary algebra homomorphism.*

**Proof.** By Corollary 2.5 there exists a unique $C^*$-ternary algebra homomorphism $H : A \to B$ such that

$$\big\|f(x) - H(x)\big\|_B \leqslant \frac{\theta}{3}\left\{\frac{1}{2-2^{p_1}}\|x\|_A^{p_1} + \frac{1}{2-2^{p_2}}\|x\|_A^{p_2} + \frac{1}{2-2^{p_3}}\|x\|_A^{p_3}\right\} \tag{4.1}$$

for all $x \in A$. It follows from (4.1) that

$$H(x) = \lim_{n\to\infty} \frac{1}{\lambda^n} f\big(\lambda^n x\big) \quad \left(H(x) = \lim_{n\to\infty} \lambda^n f\left(\frac{x}{\lambda^n}\right)\right) \tag{4.2}$$

for all $x \in A$ and all real number $\lambda > 1$ ($0 < \lambda < 1$). Therefore by the assumption, we get that $H(x_0) = e'$. Let $\lambda > 1$ and $\lim_{n\to\infty} \frac{1}{\lambda^n} f(\lambda^n x_0) = e'$. It follows from (2.12) that

$$\begin{aligned}\big\|\big[H(x), H(y), H(z)\big] - \big[H(x), H(y), f(z)\big]\big\|_B &= \big\|H[x,y,z] - \big[H(x), H(y), f(z)\big]\big\|_B \\ &= \lim_{n\to\infty} \frac{1}{\lambda^{2n}} \big\|f\big([\lambda^n x, \lambda^n y, z]\big) - \big[f\big(\lambda^n x\big), f\big(\lambda^n y\big), f(z)\big]\big\|_B \\ &\leqslant \epsilon \lim_{n\to\infty} \frac{1}{\lambda^{2n}}\big[\lambda^{nq_1}\|x\|_A^{q_1} + \lambda^{nq_2}\|y\|_A^{q_2} + \|z\|_A^{q_3})\big] = 0\end{aligned}$$

for all $x, y, z \in A$. So $[H(x), H(y), H(z)] = [H(x), H(y), f(z)]$ for all $x, y, z \in A$. Letting $x = y = x_0$ in the last equality, we get $f(z) = H(z)$ for all $z \in A$. Similarly, one can show that $H(z) = f(z)$ for all $z \in A$ when $0 < \lambda < 1$ and $\lim_{n\to\infty} \lambda^n f(\frac{x_0}{\lambda^n}) = e'$. Therefore the mapping $f : A \to B$ is a $C^*$-ternary algebra homomorphism. □

**Remark 4.2.** Theorem 4.1 will be valid if we replace the conditions $q_1, q_2 < 2$ and $q_3 < 3$ by $q_2, q_3 < 2$ and $q_1 < 3$.

**Theorem 4.3.** *Let $\epsilon, \theta, p_1, p_2, p_3, q_1, q_2$ and $q_3$ be non-negative real numbers such that $p_1, p_2, p_3 > 1$ and $q_1, q_2, q_3 > 2$. Suppose that $f : A \to B$ is a mapping satisfying* (2.11) *and*

$$\big\| f\big([x, y, z]\big) - \big[f(x), f(y), f(z)\big]\big\|_B \leqslant \epsilon\big(\|x\|_A^{q_1}\|y\|_A^{q_2} + \|y\|_A^{q_2}\|z\|_A^{q_3} + \|x\|_A^{q_1}\|z\|_A^{q_3}\big) \tag{4.3}$$

*for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. If there exist a real number $\lambda > 1$ $(0 < \lambda < 1)$ and an element $x_0 \in A$ such that $\lim_{n\to\infty} \lambda^n f(\frac{x_0}{\lambda^n}) = e'$ $(\lim_{n\to\infty} \frac{1}{\lambda^n} f(\lambda^n x_0) = e')$, then the mapping $f : A \to B$ is a $C^*$-ternary algebra homomorphism.*

**Proof.** By Theorem 2.8 there exists a unique $C^*$-ternary algebra homomorphism $H : A \to B$ such that

$$\big\| f(x) - H(x)\big\|_B \leqslant \frac{\theta}{3}\left\{\frac{1}{2^{p_1} - 2}\|x\|_A^{p_1} + \frac{1}{2^{p_2} - 2}\|x\|_A^{p_2} + \frac{1}{2^{p_3} - 2}\|x\|_A^{p_3}\right\} \tag{4.4}$$

for all $x \in A$. It follows from (4.4) that

$$H(x) = \lim_{n\to\infty} \lambda^n f\left(\frac{x}{\lambda^n}\right) \quad \left(H(x) = \lim_{n\to\infty} \frac{1}{\lambda^n} f\big(\lambda^n x\big)\right) \tag{4.5}$$

for all $x \in A$ and all real number $\lambda > 1$ $(0 < \lambda < 1)$. Therefore by the assumption, we get that $H(x_0) = e'$. Let $\lambda > 1$ and $\lim_{n\to\infty} \lambda^n f(\frac{x_0}{\lambda^n}) = e'$. It follows from (2.12) that

$$\begin{aligned}
&\big\|\big[H(x), H(y), H(z)\big] - \big[H(x), H(y), f(z)\big]\big\|_B \\
&\quad = \big\|H[x, y, z] - \big[H(x), H(y), f(z)\big]\big\|_B \\
&\quad = \lim_{n\to\infty} \lambda^{2n}\left\| f\left(\left[\frac{x}{\lambda^n}, \frac{y}{\lambda^n}, z\right]\right) - \left[f\left(\frac{x}{\lambda^n}\right), f\left(\frac{y}{\lambda^n}\right), f(z)\right]\right\|_B \\
&\quad \leqslant \epsilon \lim_{n\to\infty} \lambda^{2n}\left[\frac{1}{\lambda^{n(q_1+q_2)}}\|x\|_A^{q_1}\|y\|_A^{q_2} + \frac{1}{\lambda^{nq_2}}\|y\|_A^{q_2}\|z\|_A^{q_3} + \frac{1}{\lambda^{nq_1}}\|x\|_A^{q_1}\|z\|_A^{q_3})\right] = 0
\end{aligned}$$

for all $x, y, z \in A$. So $[H(x), H(y), H(z)] = [H(x), H(y), f(z)]$ for all $x, y, z \in A$. Letting $x = y = x_0$ in the last equality, we get $f(z) = H(z)$ for all $z \in A$. Similarly, one can show that $H(z) = f(z)$ for all $z \in A$ when $0 < \lambda < 1$ and $\lim_{n\to\infty} \frac{1}{\lambda^n} f(\lambda^n x_0) = e'$. Therefore the mapping $f : A \to B$ is a $C^*$-ternary algebra homomorphism. □

## 5. Stability of derivations on $C^*$-ternary algebras

Throughout this section, assume that $A$ is a $C^*$-ternary algebra with norm $\|\cdot\|_A$.

In this section we prove the generalized Hyers–Ulam stability of derivations on $C^*$-ternary algebras for the functional equation $D_\mu f(x, y, z) = 0$.

**Theorem 5.1.** *Let $\varphi : A^3 \to [0, \infty)$ and $\psi : A^3 \to [0, \infty)$ be functions such that*

$$\tilde{\varphi}(x) := \sum_{n=0}^{\infty} \frac{1}{2^n}\varphi\big(2^n x, 2^n x, 2^n x\big) < \infty, \qquad \lim_{n\to\infty} \frac{1}{2^n}\varphi\big(2^n x, 2^n y, 2^n z\big) = 0, \tag{5.1}$$

$$\lim_{n\to\infty} \frac{1}{8^n}\psi\big(2^n x, 2^n y, 2^n z\big) = 0 \tag{5.2}$$

*for all $x, y, z \in A$. Suppose that $f : A \to A$ is a mapping satisfying*

$$\big\|D_\mu f(x, y, z)\big\|_A \leqslant \varphi(x, y, z), \tag{5.3}$$

$$\big\| f\big([x, y, z]\big) - \big[f(x), y, z\big] - \big[x, f(y), z\big] - \big[x, y, f(z)\big]\big\|_A \leqslant \psi(x, y, z) \tag{5.4}$$

*for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. Then there exists a unique $C^*$-ternary algebra derivation $D : A \to A$ such that*

$$\left\| f(x) - D(x) \right\|_A \leqslant \frac{1}{6}\tilde{\varphi}(x) \tag{5.5}$$

*for all* $x \in A$.

**Proof.** By the proof of Theorem 2.4, there exists a unique $\mathbb{C}$-linear mapping $D : A \to A$ satisfying (5.5) and

$$D(x) := \lim_{n\to\infty} \frac{1}{2^n} f\left(2^n x\right)$$

for all $x \in A$. It follows from (5.2) and (5.4) that

$$\begin{aligned}
&\left\| D[x, y, z] - \left[D(x), y, z\right] - \left[x, D(y), z\right] - \left[x, y, D(z)\right] \right\|_A \\
&\quad = \lim_{n\to\infty} \frac{1}{8^n} \left\| f\left[2^n x, 2^n y, 2^n z\right] - \left[f\left(2^n x\right), 2^n y, 2^n z\right] - \left[2^n x, f\left(2^n y\right), 2^n z\right] - \left[2^n x, 2^n y, f\left(2^n z\right)\right] \right\|_A \\
&\quad \leqslant \lim_{n\to\infty} \frac{1}{8^n} \psi\left(2^n x, 2^n y, 2^n z\right) = 0
\end{aligned}$$

for all $x, y, z \in A$. So

$$D[x, y, z] = \left[D(x), y, z\right] + \left[x, D(y), z\right] + \left[x, y, D(z)\right]$$

for all $x, y, z \in A$. Therefore the mapping $D : A \to A$ is a $C^*$-ternary algebra derivation. □

**Theorem 5.2.** *Let* $\varphi : A^3 \to [0, \infty)$ *be a function satisfying* (5.1). *Suppose that the function* $\psi : A^3 \to [0, \infty)$ *satisfies one of the following conditions*:

(i) $\lim_{n\to\infty} \frac{1}{4^n} \psi(2^n x, 2^n y, z) = 0$;
(ii) $\lim_{n\to\infty} \frac{1}{4^n} \psi(x, 2^n y, 2^n z) = 0$;
(iii) $\lim_{n\to\infty} \frac{1}{4^n} \psi(2^n x, y, 2^n z) = 0$

*for all* $x, y, z \in A$. *Let* $f : A \to A$ *be a mapping satisfying* (5.3) *and* (5.4). *Then the mapping* $f : A \to A$ *is a* $C^*$-*ternary algebra derivation.*

**Proof.** By the proof of Theorem 2.4, there exists a $\mathbb{C}$-linear mapping $D : A \to A$ defined by

$$D(x) := \lim_{n\to\infty} \frac{1}{2^n} f\left(2^n x\right)$$

for all $x \in A$. We show that if the mapping $\psi$ satisfies one of the conditions (i), (ii) or (iii), then $f = D$.

Let $\psi$ satisfies (i) (we have a similar proof if $\psi$ satisfies (ii) or (iii)). It follows from (5.4) that

$$\begin{aligned}
&\left\| D[x, y, z] - \left[D(x), y, z\right] - \left[x, D(y), z\right] - \left[x, y, f(z)\right] \right\|_A \\
&\quad = \lim_{n\to\infty} \frac{1}{4^n} \left\| f\left[2^n x, 2^n y, z\right] - \left[f\left(2^n x\right), 2^n y, z\right] - \left[2^n x, f\left(2^n y\right), z\right] - \left[2^n x, 2^n y, f(z)\right] \right\|_A \\
&\quad \leqslant \lim_{n\to\infty} \frac{1}{4^n} \psi\left(2^n x, 2^n y, z\right) = 0
\end{aligned}$$

for all $x, y, z \in A$. Therefore

$$D\left([x, y, z]\right) = \left[D(x), y, z\right] + \left[x, D(y), z\right] + \left[x, y, f(z)\right] \tag{5.6}$$

for all $x, y, z \in A$. Replacing $z$ by $2z$ in (5.6), we get

$$2D\left([x, y, z]\right) = 2\left[D(x), y, z\right] + 2\left[x, D(y), z\right] + \left[x, y, f(2z)\right] \tag{5.7}$$

for all $x, y, z \in A$. It follows from (5.6) and (5.7) that

$$\left[x, y, f(2z) - 2f(z)\right] = 0$$

for all $x, y, z \in A$. Letting $x = y = f(2z) - 2f(z)$ in the last equation, we get

$$\|f(2z) - 2f(z)\|_A^3 = \|[f(2z) - 2f(z), f(2z) - 2f(z), f(2z) - 2f(z)]\|_A = 0$$

for all $z \in A$. So $f(2z) = 2f(z)$ for all $z \in A$. By using induction, we infer that $f(2^n z) = 2^n f(z)$ for all $z \in A$ and all $n \in \mathbb{N}$. Therefore $D(x) = f(x)$ for all $x \in A$. Hence it follows from (5.6) that the mapping $f : A \to A$ is a $C^*$-ternary derivation. $\square$

**Corollary 5.3.** *Let $\epsilon, \theta, p_1, p_2, p_3, q_1, q_2$ and $q_3$ be non-negative real numbers such that $p_1, p_2, p_3 < 1$ and $q_i < 2$ for some $1 \leqslant i \leqslant 3$. Suppose that $f : A \to A$ is a mapping satisfying*

$$\|D_\mu f(x, y, z)\|_A \leqslant \theta\big(\|x\|_A^{p_1} + \|y\|_A^{p_2} + \|z\|_A^{p_3}\big), \tag{5.8}$$

$$\big\|f\big([x, y, z]\big) - \big[f(x), y, z\big] - \big[x, f(y), z\big] - \big[x, y, f(z)\big]\big\|_A \leqslant \epsilon\big(\|x\|_A^{q_1} + \|y\|_A^{q_2} + \|z\|_A^{q_3}\big) \tag{5.9}$$

*for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. Then the mapping $f : A \to A$ is a $C^*$-ternary algebra derivation.*

**Theorem 5.4.** *Let $\varphi : A^3 \to [0, \infty)$ be a function satisfying (5.1). Suppose that the function $\psi : A^3 \to [0, \infty)$ satisfies one of the following conditions:*

(i) $\lim_{n\to\infty} \frac{1}{2^n}\psi(2^n x, y, z) = 0$;
(ii) $\lim_{n\to\infty} \frac{1}{2^n}\psi(x, 2^n y, z) = 0$;
(iii) $\lim_{n\to\infty} \frac{1}{2^n}\psi(x, y, 2^n z) = 0$

*for all $x, y, z \in A$. Let $f : A \to A$ be a mapping satisfying (5.3) and (5.4). Then the mapping $f : A \to A$ is a $C^*$-ternary algebra derivation.*

**Proof.** By the proof of Theorem 2.4, there exists a $\mathbb{C}$-linear mapping $D : A \to A$ defined by

$$D(x) := \lim_{n\to\infty} \frac{1}{2^n} f\big(2^n x\big)$$

for all $x \in A$. We show that if the mapping $\psi$ satisfies one of the conditions (i), (ii) or (iii), then $f = D$.

Let $\psi$ satisfies (i) (we have a similar proof if $\psi$ satisfies (ii) or (iii)). It follows from (5.4) that

$$\begin{aligned}
&\big\|D[x, y, z] - \big[D(x), y, z\big] - \big[x, f(y), z\big] - \big[x, y, f(z)\big]\big\|_A \\
&\quad = \lim_{n\to\infty} \frac{1}{2^n}\big\|f\big[2^n x, y, z\big] - \big[f(2^n x), y, z\big] - \big[2^n x, f(y), z\big] - \big[2^n x, y, f(z)\big]\big\|_A \\
&\quad \leqslant \lim_{n\to\infty} \frac{1}{2^n}\psi\big(2^n x, y, z\big) = 0
\end{aligned}$$

for all $x, y, z \in A$. Therefore

$$D\big([x, y, z]\big) = \big[D(x), y, z\big] + \big[x, f(y), z\big] + \big[x, y, f(z)\big] \tag{5.10}$$

for all $x, y, z \in A$.

The rest of the proof is similar to the proof Theorem 5.2. $\square$

**Theorem 5.5.** *Let $\Phi : A^3 \to [0, \infty)$ and $\Psi : A^3 \to [0, \infty)$ be functions such that*

$$\widetilde{\Phi}(x) := \sum_{n=1}^{\infty} 2^n \Phi\left(\frac{x}{2^n}, \frac{x}{2^n}, \frac{x}{2^n}\right) < \infty, \qquad \lim_{n\to\infty} 2^n \Phi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) = 0, \tag{5.11}$$

$$\lim_{n\to\infty} 8^n \Psi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) = 0 \tag{5.12}$$

*for all $x, y, z \in A$. Suppose that $f : A \to A$ is a mapping satisfying*

$$\left\| D_\mu f(x,y,z) \right\|_A \leqslant \Phi(x,y,z), \tag{5.13}$$

$$\left\| f([x,y,z]) - [f(x),y,z] - [x,f(y),z] - [x,y,f(z)] \right\|_A \leqslant \Psi(x,y,z) \tag{5.14}$$

*for all $\mu \in \mathbb{T}^1$ and all $x,y,z \in A$. Then there exists a unique $C^*$-ternary algebra derivation $D : A \to A$ such that*

$$\left\| f(x) - D(x) \right\|_A \leqslant \frac{1}{6} \widetilde{\Phi}(x) \tag{5.15}$$

*for all $x \in A$.*

**Proof.** By the proof of Theorem 2.8, there exists a unique $\mathbb{C}$-linear mapping $D : A \to A$ satisfying (5.15) and

$$D(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in A$.

The rest of the proof is similar to the proof of Theorem 5.1. □

**Theorem 5.6.** *Let $\Phi : A^3 \to [0,\infty)$ be a function satisfying* (5.11). *Suppose that the function $\Psi : A^3 \to [0,\infty)$ satisfies one of the following conditions*:

(i) $\lim_{n\to\infty} 4^n \Psi(\frac{x}{2^n}, \frac{y}{2^n}, z) = 0$;
(ii) $\lim_{n\to\infty} 4^n \Psi(x, \frac{y}{2^n}, \frac{z}{2^n}) = 0$;
(iii) $\lim_{n\to\infty} 4^n \Psi(\frac{x}{2^n}, y, \frac{z}{2^n}) = 0$

*for all $x,y,z \in A$. Let $f : A \to A$ be a mapping satisfying* (5.13) *and* (5.14). *Then the mapping $f : A \to A$ is a $C^*$-ternary algebra derivation.*

**Proof.** By the proof of Theorem 2.8, there exists a $\mathbb{C}$-linear mapping $D : A \to A$ defined by

$$D(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in A$.

The rest of the proof is similar to the proof of Theorem 5.2. □

**Corollary 5.7.** *Let $\epsilon, \theta, p_1, p_2, p_3, q_1, q_2$ and $q_3$ be non-negative real numbers such that $p_1, p_2, p_3 > 1$ and $q_1, q_2, q_3 > 2$. Suppose that $f : A \to A$ is a mapping satisfying* (5.8) *and*

$$\left\| f([x,y,z]) - [f(x),y,z] - [x,f(y),z] - [x,y,f(z)] \right\|_A \leqslant \epsilon\left(\|x\|_A^{q_1}\|y\|_A^{q_2} + \|y\|_A^{q_2}\|z\|_A^{q_3} + \|x\|_A^{q_1}\|z\|_A^{q_3}\right) \tag{5.16}$$

*for all $x,y,z \in A$. Then the mapping $f : A \to A$ is a $C^*$-ternary algebra derivation.*

**Theorem 5.8.** *Let $\Phi : A^3 \to [0,\infty)$ be a function satisfying* (5.11). *Suppose that the function $\Psi : A^3 \to [0,\infty)$ satisfies one of the following conditions*:

(i) $\lim_{n\to\infty} 2^n \Psi(\frac{x}{2^n}, y, z) = 0$;
(ii) $\lim_{n\to\infty} 2^n \Psi(x, \frac{y}{2^n}, z) = 0$;
(iii) $\lim_{n\to\infty} 2^n \Psi(x, y, \frac{z}{2^n}) = 0$

*for all $x,y,z \in A$. Let $f : A \to A$ be a mapping satisfying* (5.13) *and* (5.14). *Then the mapping $f : A \to A$ is a $C^*$-ternary algebra derivation.*

**Proof.** By the proof of Theorem 2.8, there exists a $\mathbb{C}$-linear mapping $D : A \to A$ defined by

$$D(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in A$.

The rest of the proof is similar to the proof of Theorem 5.4. $\square$

**Theorem 5.9.** *Let $\epsilon, \theta, p_1, p_2, p_3$ be non-negative real numbers and let $q_1, q_2, q_3$ be real numbers such that $p_i > 0$ and $q_j \neq 1$ for some $1 \leqslant i, j \leqslant 3$. Suppose that $f : A \to A$ is a mapping satisfying*

$$\left\|D_\mu f(x, y, z)\right\|_A \leqslant \theta \|x\|_A^{p_1} \|y\|_A^{p_2} \|z\|_A^{p_3}, \tag{5.17}$$

$$\left\|f\big([x, y, z]\big) - \big[f(x), y, z\big] - \big[x, f(y), z\big] - \big[x, y, f(z)\big]\right\|_A \leqslant \epsilon \|x\|_A^{q_1} \|y\|_A^{q_2} \|z\|_A^{q_3} \tag{5.18}$$

*for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$ ($x, y, z \in A \setminus \{0\}$ when $q_i < 0$ for some $1 \leqslant i \leqslant 3$). Then the mapping $f : A \to A$ is a $C^*$-ternary algebra derivation.*

**Proof.** Without any loss of generality, we may assume that $q_1 \neq 1$ and $p_1 > 0$. Therefore it follows from the proof of Theorem 3.4 that the mapping $f : A \to A$ is $\mathbb{C}$-linear. Let $q_1 < 1$. It follows from (5.18) that

$$\begin{aligned}
&\left\|f\big([x, y, z]\big) - \big[f(x), y, z\big] - \big[x, f(y), z\big] - \big[x, y, f(z)\big]\right\|_A \\
&\quad = \lim_{n\to\infty} \frac{1}{2^n} \left\|f\big[2^n x, y, z\big] - \big[f\big(2^n x\big), y, z\big] - \big[2^n x, f(y), z\big] - \big[2^n x, y, f(z)\big]\right\|_A \\
&\quad \leqslant \epsilon \lim_{n\to\infty} \frac{2^{nq_1}}{2^n} \|x\|_A^{q_1} \|y\|_A^{q_2} \|z\|_A^{q_3} = 0
\end{aligned}$$

for all $x, y, z \in A$ ($x, y, z \in A \setminus \{0\}$ when $q_i < 0$ for some $1 \leqslant i \leqslant 3$). Therefore

$$f\big([x, y, z]\big) = \big[f(x), y, z\big] + \big[x, f(y), z\big] + \big[x, y, f(z)\big] \tag{5.19}$$

for all $x, y, z \in A$ ($x, y, z \in A \setminus \{0\}$ when $q_i < 0$ for some $1 \leqslant i \leqslant 3$). Since $f(0) = 0$, then (5.20) holds for all $x, y, z \in A$ when $q_i < 0$ for some $1 \leqslant i \leqslant 3$. Similarly, we get (5.20) when $q_1 > 1$. So the mapping $f : A \to B$ is a $C^*$-ternary algebra derivation. $\square$

**Theorem 5.10.** *Let $q_1, q_2, q_3$ be real numbers and let $\epsilon, \theta, p_1, p_2, p_3$ be non-negative real numbers such that $p_i > 0$ and $q_1 + q_2 + q_3 \neq 3$ for some $1 \leqslant i \leqslant 3$. Suppose that $f : A \to A$ is a mapping satisfying* (5.17) *and* (5.18). *Then the mapping $f : A \to B$ is a $C^*$-ternary algebra derivation.*

**Proof.** It follows from the proof of Theorem 3.4 that the mapping $f : A \to A$ is $\mathbb{C}$-linear. Let $q_1 + q_2 + q_3 < 3$. It follows from (5.18) that

$$\begin{aligned}
&\left\|f\big([x, y, z]\big) - \big[f(x), y, z\big] - \big[x, f(y), z\big] - \big[x, y, f(z)\big]\right\|_A \\
&\quad = \lim_{n\to\infty} \frac{1}{8^n} \left\|f\big(\big[2^n x, 2^n y, 2^n z\big]\big) - \big[f\big(2^n x\big), 2^n y, 2^n z\big] - \big[2^n x, f\big(2^n y\big), 2^n z\big] - \big[2^n x, 2^n y, f\big(2^n z\big)\big]\right\|_A \\
&\quad \leqslant \epsilon \lim_{n\to\infty} \frac{2^{n(q_1+q_2+q_3)}}{8^n} \|x\|_A^{q_1} \|y\|_A^{q_2} \|z\|_A^{q_3} = 0
\end{aligned}$$

for all $x, y, z \in A$ ($x, y, z \in A \setminus \{0\}$ when $q_i < 0$ for some $1 \leqslant i \leqslant 3$). Therefore

$$f\big([x, y, z]\big) = \big[f(x), y, z\big] + \big[x, f(y), z\big] + \big[x, y, f(z)\big] \tag{5.20}$$

for all $x, y, z \in A$ ($x, y, z \in A \setminus \{0\}$ when $q_i < 0$ for some $1 \leqslant i \leqslant 3$). Since $f(0) = 0$, then (5.20) holds for all $x, y, z \in A$ when $q_i < 0$ for some $1 \leqslant i \leqslant 3$. Similarly, we get (5.20) when $q_1 + q_2 + q_3 > 3$. So the mapping $f : A \to B$ is a $C^*$-ternary algebra derivation. $\square$

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## References

[1] M. Amyari, M.S. Moslehian, Approximately ternary semigroup homomorphisms, Lett. Math. Phys. 77 (2006) 1–9.
[2] T. Aoki, On the stability of the linear transformation in Banach spaces, J. Math. Soc. Japan 2 (1950) 64–66.
[3] P. Czerwik, Functional Equations and Inequalities in Several Variables, World Scientific Publishing Company, New Jersey, 2002.
[4] Z. Gajda, On stability of additive mappings, Int. J. Math. Math. Sci. 14 (1991) 431–434.
[5] P. Găvruta, A generalization of the Hyers–Ulam–Rassias stability of approximately additive mappings, J. Math. Anal. Appl. 184 (1994) 431–436.
[6] P. Găvruta, An answer to a question of John Rassias concerning the stability of Cauchy equation, in: Advances in Equations and Inequalities, in: Hadronic Math. Ser., USA, 1999, pp. 67–71.
[7] D.H. Hyers, On the stability of the linear functional equation, Proc. Natl. Acad. Sci. USA 27 (1941) 222–224.
[8] D.H. Hyers, G. Isac, Th.M. Rassias, Stability of Functional Equations in Several Variables, Birkhäuser, Basel, 1998.
[9] D.H. Hyers, G. Isac, Th.M. Rassias, On the asymptoticity aspect of Hyers–Ulam stability of mappings, Proc. Amer. Math. Soc. 126 (1998) 425–430.
[10] G. Isac, Th.M. Rassias, Stability of $\psi$-additive mappings: Applications to nonlinear analysis, Int. J. Math. Math. Sci. 19 (1996) 219–228.
[11] K.W. Jun, H.-M. Kim, On the stability of Euler–Lagrange type cubic mappings in quasi-Banach spaces, J. Math. Anal. Appl. 332 (2007) 1334–1349.
[12] Soon-Mo Jung, Hyers–Ulam–Rassias stability of Jensen's equation and its application, Proc. Amer. Math. Soc. 126 (1998) 3137–3143.
[13] S. Jung, On the Hyers–Ulam–Rassias stability of approximately additive mappings, J. Math. Anal. Appl. 204 (1996) 221–226.
[14] Y.S. Jung, On the generalized Hyers–Ulam stability of module left derivations, J. Math. Anal. Appl. (2007), doi:10.1016/j.jmaa.2007.07.003.
[15] C. Park, On the stability of the linear mapping in Banach modules, J. Math. Anal. Appl. 275 (2002) 711–720.
[16] C. Park, Approximate homomorphisms on $JB^*$-triples, J. Math. Anal. Appl. 306 (2005) 375–381.
[17] C. Park, Homomorphisms between Poisson $JC^*$-algebras, Bull. Braz. Math. Soc. 36 (2005) 79–97.
[18] C. Park, Isomorphisms between unital $C^*$-algebras, J. Math. Anal. Appl. 307 (2005) 753–762.
[19] C. Park, Hyers–Ulam–Rassias stability of homomorphisms in quasi-Banach algebras, Bull. Sci. Math. (2006), doi:10.1016/j.bulsci.2006.07.004.
[20] C. Park, Lie $*$-homomorphisms between Lie $C^*$-algebras and Lie $*$-derivations on Lie $C^*$-algebras, J. Math. Anal. Appl. 293 (2004) 419–434.
[21] C. Park, Homomorphisms between Lie $JC^*$-algebras and Cauchy–Rassias stability of Lie $JC^*$-algebra derivations, J. Lie Theory 15 (2005) 393–414.
[22] C. Park, Y. Cho, M. Han, Stability of functional inequalities associated with Jordan–von Neumann type additive functional equations, J. Inequal. Appl. 2007 (41820) (2007) 1–12.
[23] C. Park, J. Hou, S. Oh, Homomorphisms between $JC^*$-algebras and between Lie $C^*$-algebras, Acta Math. Sinica 21 (2005) 1391–1398.
[24] Ch. Park, A. Najati, Homomorphisms and derivations in $C^*$-algebras, Abstr. Appl. Anal. 2007 (2007), Article ID 80630.
[25] J.M. Rassias, On approximation of approximately linear mappings by linear mappings, J. Funct. Anal. 46 (1) (1982) 126–130.
[26] J.M. Rassias, On approximation of approximately linear mappings by linear mappings, Bull. Sci. Math. 108 (1984) 445–446.
[27] J.M. Rassias, Solution of a problem of Ulam, J. Approx. Theory 57 (3) (1989) 268–273.
[28] J.M. Rassias, Solution of a stability problem of Ulam, Discuss. Math. 12 (1992) 95–103.
[29] J.M. Rassias, Alternative contraction principle and alternative Jensen and Jensen type mappings, Int. J. Appl. Math. Stat. 2 (2005) 92–101.
[30] J.M. Rassias, On the Cauchy–Ulam stability of the Jensen equation in $C^*$-algebras, Int. J. Pure Appl. Math. Stat. 4 (5) (2006) 1–10.
[31] J.M. Rassias, Refined Hyers–Ulam approximation of approximately Jensen type mappings, Bull. Sci. Math. 131 (1) (2007) 89–98.
[32] Th.M. Rassias, On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. 72 (1978) 297–300.
[33] Th.M. Rassias, Problem 16; 2, Report of the 27th International Symp. on Functional Equations, Aequationes Math. 39 (1990) 292–293, 309.
[34] Th.M. Rassias, The problem of S.M. Ulam for approximately multiplicative mappings, J. Math. Anal. Appl. 246 (2000) 352–378.
[35] Th.M. Rassias, On the stability of functional equations in Banach spaces, J. Math. Anal. Appl. 251 (2000) 264–284.
[36] Th.M. Rassias, Functional Equations, Inequalities and Applications, Kluwer Academic Publishers, Dordrecht, 2003.
[37] Th.M. Rassias, P. Šemrl, On the behaviour of mappings which do not satisfy Hyers–Ulam stability, Proc. Amer. Math. Soc. 114 (1992) 989–993.
[38] F. Skof, Proprietà locali e approssimazione di operatori, Rend. Sem. Mat. Fis. Milano 53 (1983) 113–129.
[39] S.M. Ulam, A Collection of the Mathematical Problems, Interscience Publ., New York, 1960.
[40] H. Zettl, A characterization of ternary rings of operators, Adv. Math. 48 (1983) 117–143.